Fourier Transform Properties

- Fourier transform pairs & notation
- Linearity
- Scaling
- Amplitude and phase spectra
- Conjugate operation
- Symmetry
- Signal „area”
- Shifting in time
- Shifting in frequency
- Modulation
- Convolution in time
- Signal filtering (frequency & time domain)
- Pulse shaping (+)

Fourier transform & notation

TRANSFORM

FORWARD

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$$

$$X(\omega) = \mathcal{F}\{x(t)\}$$

INVERSE

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

TRANSFORM PAIRS

$$x(t) \leftrightarrow X(\omega)$$

Linearity

$$\mathcal{F}\{\alpha x(t)+\beta y(t)\}=\mathcal{F}\{\alpha x(t)\}+\mathcal{F}\{\beta y(t)\}=$$

$$\alpha\mathcal{F}\{x(t)\}+\beta\mathcal{F}\{y(t)\}=\alpha X(\omega)+\beta Y(\omega)$$

Transforms of complicated signals may be found piece-by-piece.

Scaling

$$x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{\omega}{\alpha}\right)$$

- **Squeezing signal in the time domain (*α* > 1) broadens its spectrum; extending a signal (*α* < 1) narrows its spectrum.**
- **The shorter the signal is the broader its spectrum is.**

$$\mathbf{1}(t)\mathrm{e}^{-t} \leftrightarrow (1 + j\omega)^{-1} = \frac{1}{1 + j\omega}$$

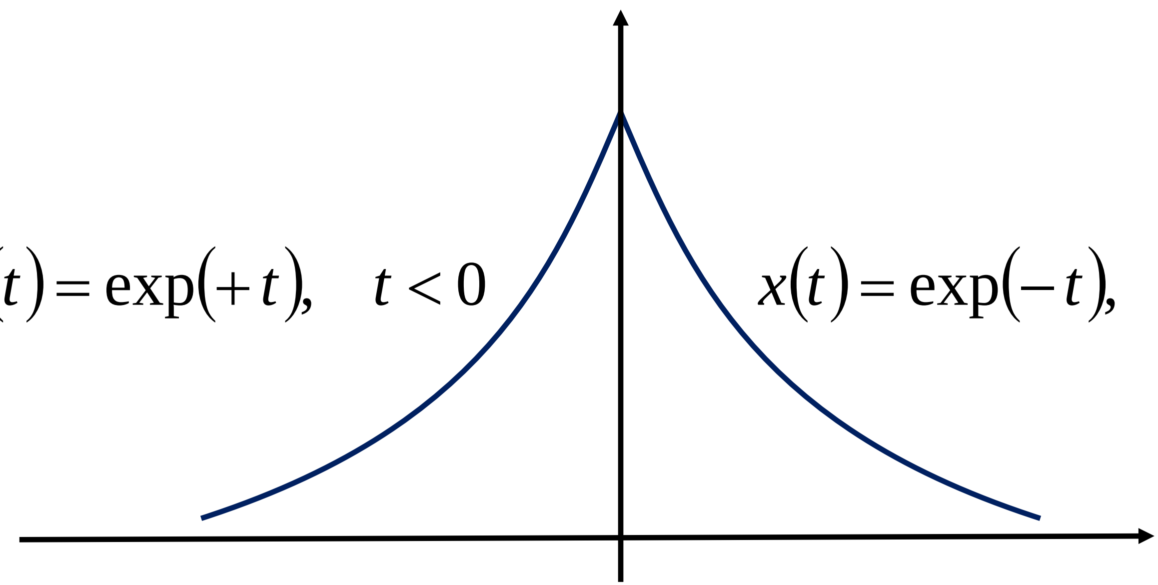
$$\mathbf{1}(t)\mathrm{e}^{-\alpha t} \leftrightarrow (1 + j\,\omega/\alpha)^{-1}/\alpha = \frac{1}{\alpha + j\omega}, \alpha > 0$$

Prove:

Two-sided exponential signal

Find Fourier transform of the two-sided exponential signal (Laplace distribution):

$$x(t) = \exp(+t), \quad t < 0 \qquad\qquad x(t) = \exp(-t), \quad t \geq 0$$



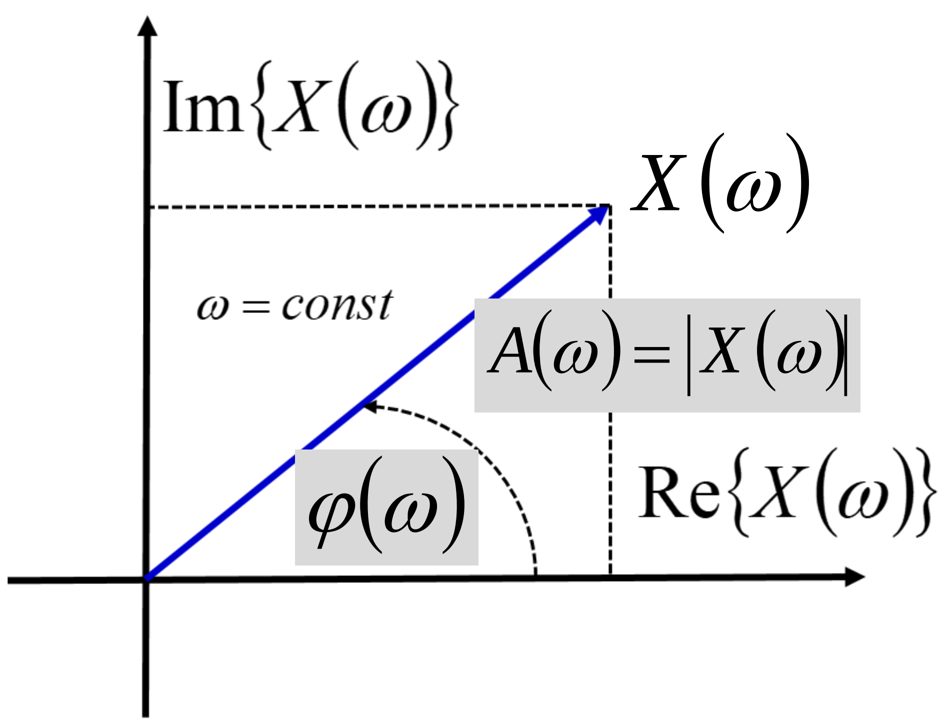
Amplitude & phase spectra

$$x(t) \leftrightarrow X(\omega) = A(\omega)e^{j\varphi(\omega)} = |X(\omega)|e^{j\varphi(\omega)}$$

exponential representation of a complex number

$A(\omega) = |X(\omega)|$ - amplitude - frequency spectrum

$\mathrm{tg}\,\varphi(\omega) = \mathrm{Im}\{X(\omega)\}/\mathrm{Re}\{X(\omega)\}$ - phase - frequency spectrum



Amplitude & phase spectra

Prove that due to Hermite symmetry a-f and p-f follow symmetry:

$$A(\omega) = A(-\omega) \text{ - a - f characteristic is an even function}$$

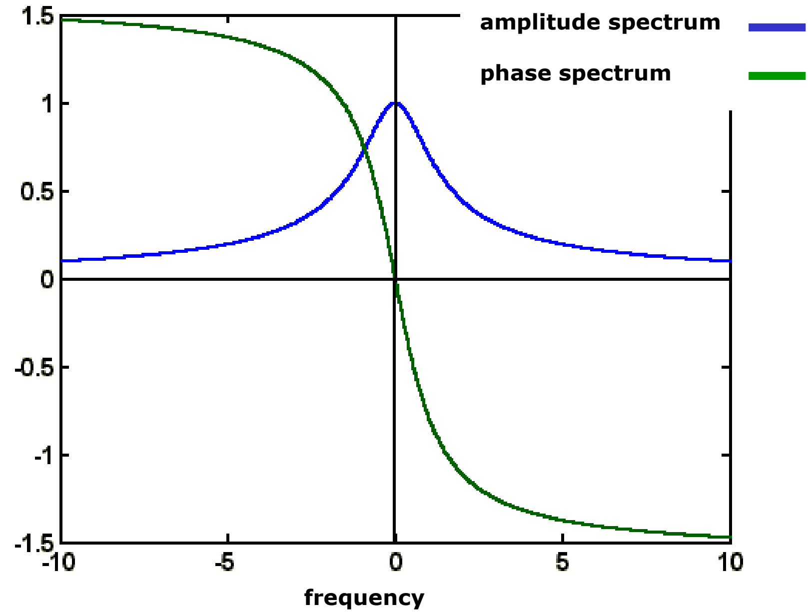
$$\varphi(\omega) = -\varphi(-\omega) \text{ - p - f characteristic is an odd function}$$

Exponential signal – a-f and p-f spectra

$$\mathbf{1}(t)e^{-t} \leftrightarrow \frac{1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2} = \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2}$$

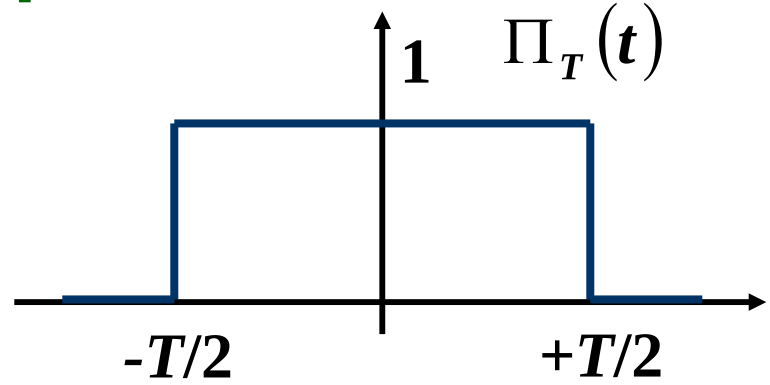
$$A(\omega) = 1 \big/ \sqrt{1+\omega^2}$$

$$\varphi(\omega) = -\mathrm{arctg}(\omega)$$



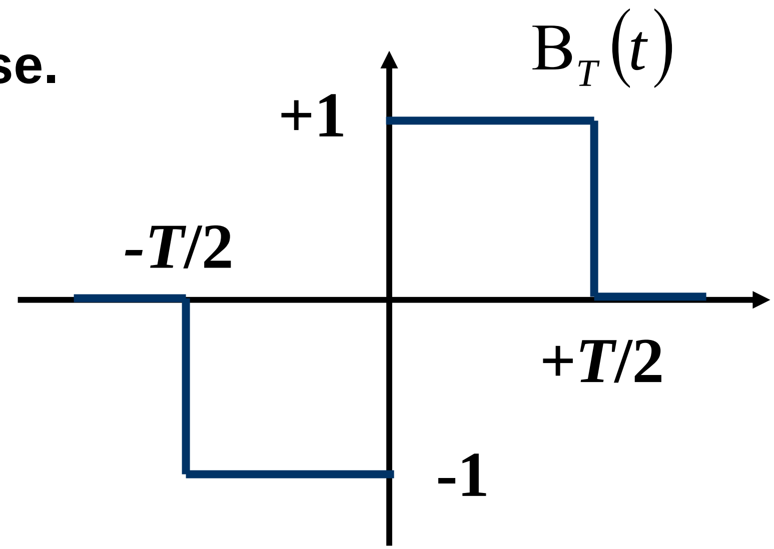
Bipulse signal – a-f spectrum

$$\Pi_T(t) \leftrightarrow T\,\mathrm{Sa}\,\frac{\omega T}{2}$$



rectangular pulse

Find Fourier transform of the bipulse. Compare a-f spectra of both pulses on a common graph.



bipulse

Conjugate operation

$$x^*(t) \leftrightarrow X^*(-\omega)$$

Prove

$$\mathcal{F}\left\{x^*(t)\right\} \overset{\text{df}}{=} \ldots$$

Hermite symmetry for real signals (test for correct calculations):

$$x(t) \in \mathcal{R},\; x(t) = x^*(t)$$

$$X(\omega) = X^*(-\omega)$$

Time reversal in frequency domain

Prove

$$x(-t) \leftrightarrow X^{*}(\omega)$$
$$x(t) \in \mathcal{R}$$

Time reversal is represented in the frequency domain by the conjugate operation.

Symmetry (duality)

$$x(t) \leftrightarrow X(\omega)$$

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Fourier transform pair may be easily converted to another transform pair.

$$\Pi_T(t) \leftrightarrow T\mathrm{Sa}(\omega T/2) \text{ where } \mathrm{Sa}(x) = \sin x/x$$

Prove

$$T\mathrm{Sa}(tT/2) \leftrightarrow 2\pi\,\Pi_T(-\omega) \qquad T/2 = W$$

$$\mathrm{Sa}(Wt) \leftrightarrow \frac{\pi}{W}\Pi_{2W}(\omega)$$

$$\mathrm{Sa}(Wt) \leftrightarrow \frac{\pi}{W}\Pi_{2W}(\omega)$$

Symmetry (duality)

Prove

$$x(t) \leftrightarrow X(\omega)$$
$$X(t) \leftrightarrow 2\pi x(-\omega)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \quad \left| t \rightarrow -t \right.$$

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega \quad \left| \times 2\pi \right.$$

$$2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)e^{-j\omega t}d\omega \quad \left| \omega \leftrightarrow t \right.$$

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)e^{-j\omega t}dt$$

$$X(t) \leftrightarrow 2\pi x(-\omega)$$

Signal ‘area’
(dc component of the signal)
(dc – direct current)

$$\int_{-\infty}^{+\infty} x(t)dt = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\Big|_{\omega=0} = X(\omega=0) = X(0)$$

Signal „area” is equal to the value of its Fourier transform at *ω* = 0.

Prove

$$\mathrm{Sa}(Wt) = \frac{\sin Wt}{Wt} \leftrightarrow \frac{\pi}{W}\Pi_{2W}(\omega)$$

$$\int_{-\infty}^{+\infty} \frac{\sin Wt}{Wt}dt = \frac{\pi}{W}\Pi_{2W}(\omega=0) = \frac{\pi}{W}$$

Shifting in time

Prove:

$$x(t-\tau) \leftrightarrow X(\omega)e^{-j\omega\tau}$$

Shifting in time modifies a phase spectrum:

$$e^{-j\omega\tau}X(\omega) = e^{-j\omega\tau}A(\omega)e^{j\varphi(\omega)}$$
$$= A(\omega)e^{j[\varphi(\omega)-\omega\tau]}$$

Amplitude spectrum is unchanged.

Shifting in frequency

Prove

$$x(t)e^{\pm j\omega_0 t} \leftrightarrow X(\omega \mp \omega_0)$$

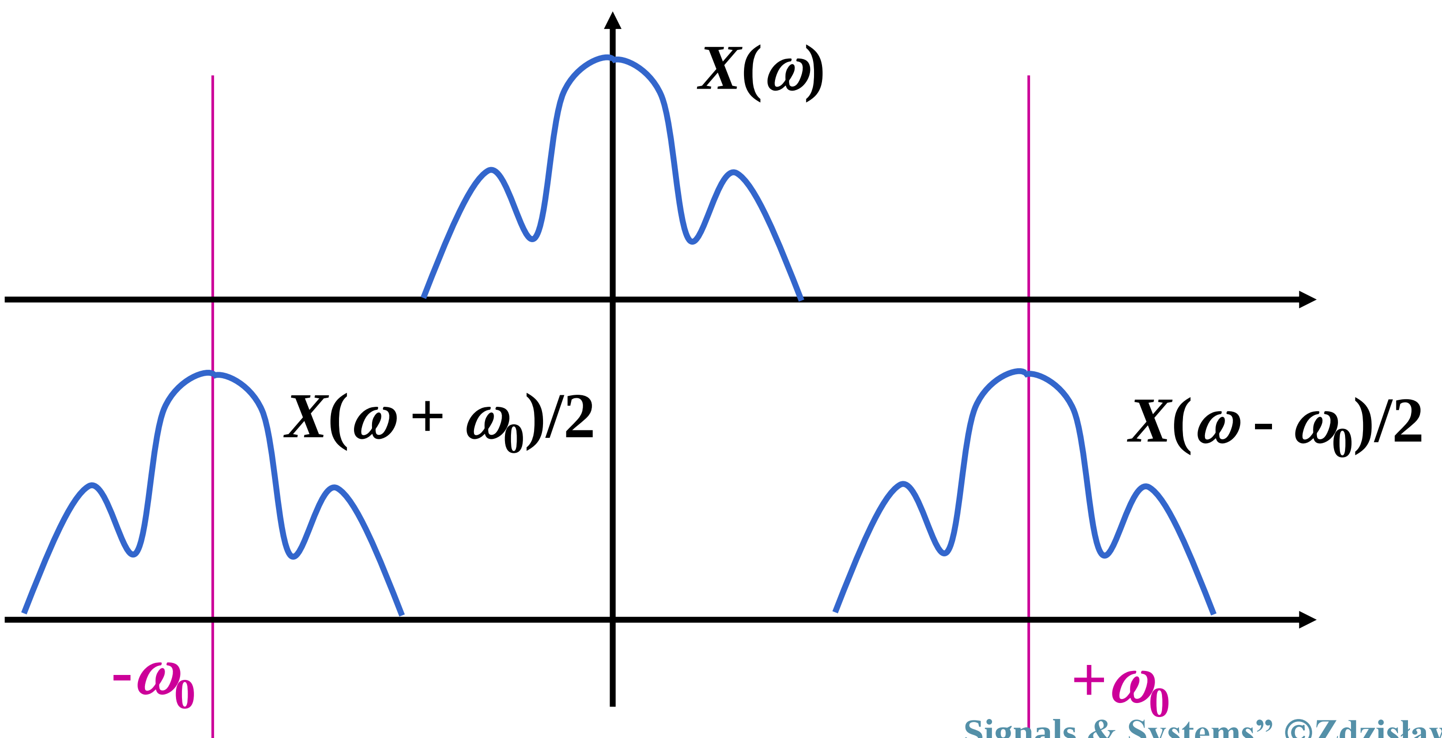
Shifting spectrum in the frequency domain is caused by multiplication in the time domain by a factor $e^{\pm j\omega_0 t}$

Shifting in frequency property generates the famous <u>modulation property</u>.

Modulation

$$x(t)\exp(+j\omega_0 t) \leftrightarrow X(\omega-\omega_0)$$
$$x(t)\exp(-j\omega_0 t) \leftrightarrow X(\omega+\omega_0)$$
$$x(t)\cos\omega_0 t \leftrightarrow \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]$$

Modulation implemented in the time domain shifts the signal spectrum from its natural position to desired higher frequencies (channel bandwidth).



Convolution in time

$$x(t)*y(t)=\int_{-\infty}^{+\infty}x(\tau)y(t-\tau)d\tau$$

PROPERTIES

Commutation $$x(t)*y(t)=y(t)*x(t)$$

Prove

Association $$[x(t)*y(t)]*z(t)=x(t)*[y(t)*z(t)]$$

Distribution with addition $$x(t)*[y(t)+z(t)]=$$ $$=x(t)*y(t)+x(t)*z(t)$$

Convolution in time

$$x(t)*y(t)=\int_{-\infty}^{+\infty}x(\tau)y(t-\tau)d\tau$$

$$\boxed{x(t)*y(t)\leftrightarrow X(\omega)Y(\omega)}$$

Difficult operation (convolution) in the time domain is replaced by a common product of transforms in the frequency domain.

Using the definition of a convolution in the time domain and its spectral representation find the signal $x(t)$ given by the formula:

$$x(t)=\frac{W}{\pi}\int_{-\infty}^{+\infty}\mathrm{Sa}(W\tau)\cos\omega_0(t-\tau)d\tau.$$

Prove

Convolution in time

$$x(t)*y(t)\leftrightarrow \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(t-\tau)y(\tau)d\tau\right]e^{-j\omega t}dt=$$

$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}x(t-\tau)e^{-j\omega t}dt\right]y(\tau)d\tau=$$

time shift

$$=\int_{-\infty}^{\infty}X(\omega)e^{-j\omega\tau}y(\tau)d\tau=$$

$$=X(\omega)\int_{-\infty}^{\infty}y(\tau)e^{-j\omega\tau}d\tau=X(\omega)Y(\omega)$$

Signal filtering in frequency domain

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

$$\frac{1}{2\pi}X(\omega)e^{j\omega t} \quad x(t) \longrightarrow \boxed{H(s:=j\omega)} \longrightarrow y(t) \quad \frac{1}{2\pi}H(\omega)X(\omega)e^{j\omega t}$$

Fourier transform of signal *y*(*t*)

$$y(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}H(\omega)X(\omega)e^{j\omega t}d\omega$$

$$Y(\omega)=H(\omega)X(\omega)$$

Fourier transform of an output filter signal *Y*(*ω*) is equal to a product of a filter transfer function *H*(*ω*) and a Fourier transform of an input filter signal *X*(*ω*).

Signal filtering in time domain

$$x(t) \leftrightarrow X(\omega) \longrightarrow \boxed{H(s := j\omega)} \longrightarrow Y(\omega) = H(\omega)X(\omega)$$

$$Y(\omega) = H(\omega)X(\omega) \leftrightarrow y(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

$$h(t) \leftrightarrow H(\omega) - \text{impulse response of a filter}$$

Output signal filter *y*(*t*) is equal to a convolution of an impulse response of a filter *h*(*t*) → *H*(*ω*) with an input signal *x*(*t*).

Signal filtering

$$x(t) \leftrightarrow X(\omega) \longrightarrow \boxed{H(s := j\omega)} \longrightarrow Y(\omega) = H(\omega)X(\omega)$$

$$X(\omega) = |X(\omega)|e^{j\varphi(\omega)}$$

$$H(\omega) = |H(\omega)|e^{j\gamma(\omega)}$$

$$Y(\omega) = |X(\omega)||H(\omega)|e^{j[\varphi(\omega)+\gamma(\omega)]}$$

Signal filtering modifies both:

- **a-f spectrum**
- **p-f spectrum**

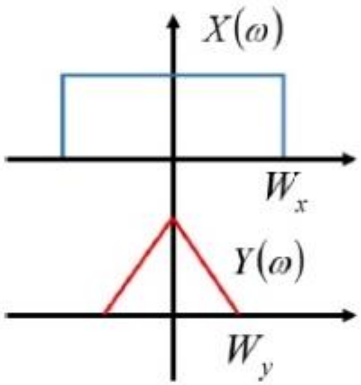
of an input signal.

Convolution in frequency

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * Y(\omega)$$

$$x(t)y(t) \leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\nu) Y(\omega - \nu) d\nu$$

Prove



Find the bandwidth of a product $x(t)y(t)$ of two lowband signals $x(t)$ and $y(t)$:

$$x(t) \leftrightarrow X(\omega),\ X(\omega) = 0 \text{ for } |\omega| \geq W_x$$

$$y(t) \leftrightarrow Y(\omega),\ Y(\omega) = 0 \text{ for } |\omega| \geq W_y$$

1. Let signal $x(t)$ be lowband with a bandwidth $W << \omega_0$. What is the minimum and maximum frequency of a signal $x^2(t)\cos\omega_0 t$?

2. Let signal $x(t)$ be lowband with a bandwidth W. What is the bandwidth of a signal $\sqrt{x(t)}$?

Pulse shaping

Riemann–Lebesgue lemma:

Fourier transform $\to 0$ with frequency $\to \infty$.

$$\lim_{\omega\to\infty} X(\omega) = 0$$

Decaying rate of Fourier transform (for *T*-pulses) is:

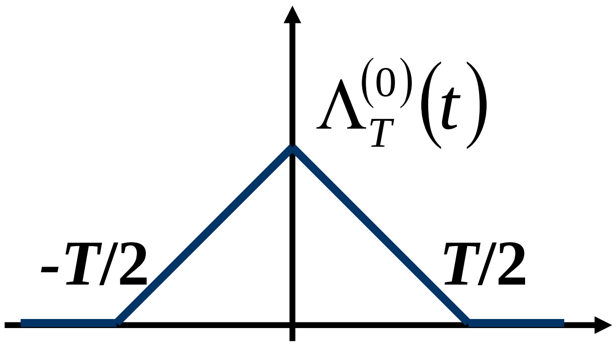
$$\lim_{|\omega|\to\infty} 1/\omega^{n+2} = 0 \qquad -20\log\omega^{n+2} = -20(n+2)[\mathrm{dB/dek}]$$

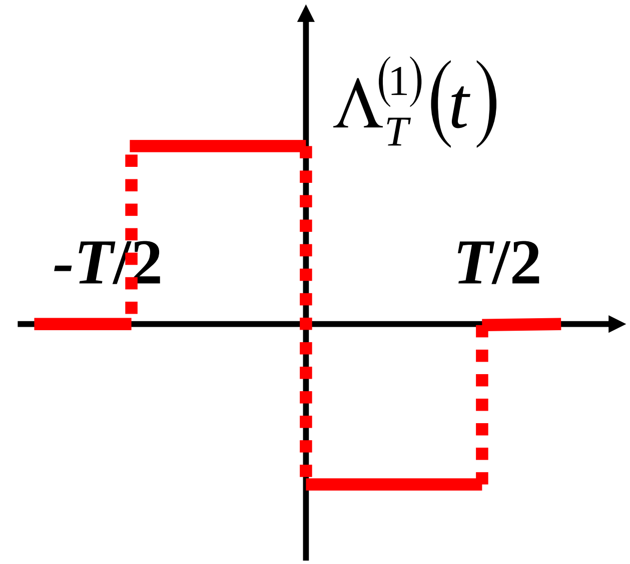
$$\lim_{|\omega|\to\infty} \frac{X(\omega)}{1/\omega^{n+2}} = \lim_{|\omega|\to\infty} \omega^{n+2} X(\omega) = const \neq 0$$

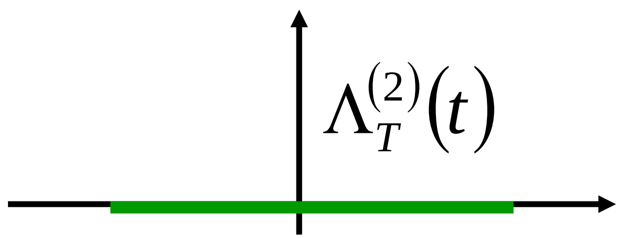
provided continuous derivatives exist: $x(\pm T/2)\ldots x^{(n)}(\pm T/2)$

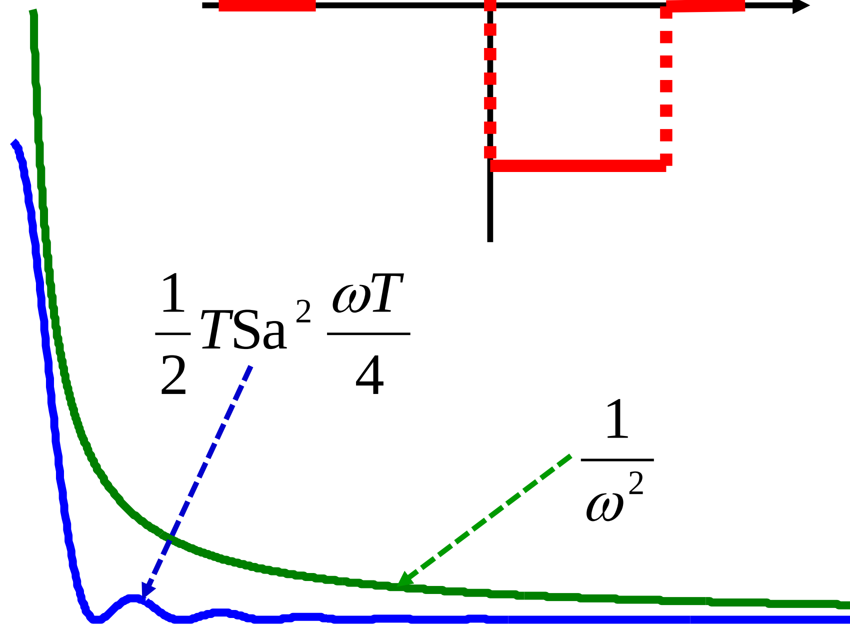
The smoother is the signal at its edge points ∓*T*/2, the faster its spectrum decays to zero.

Pulse shaping





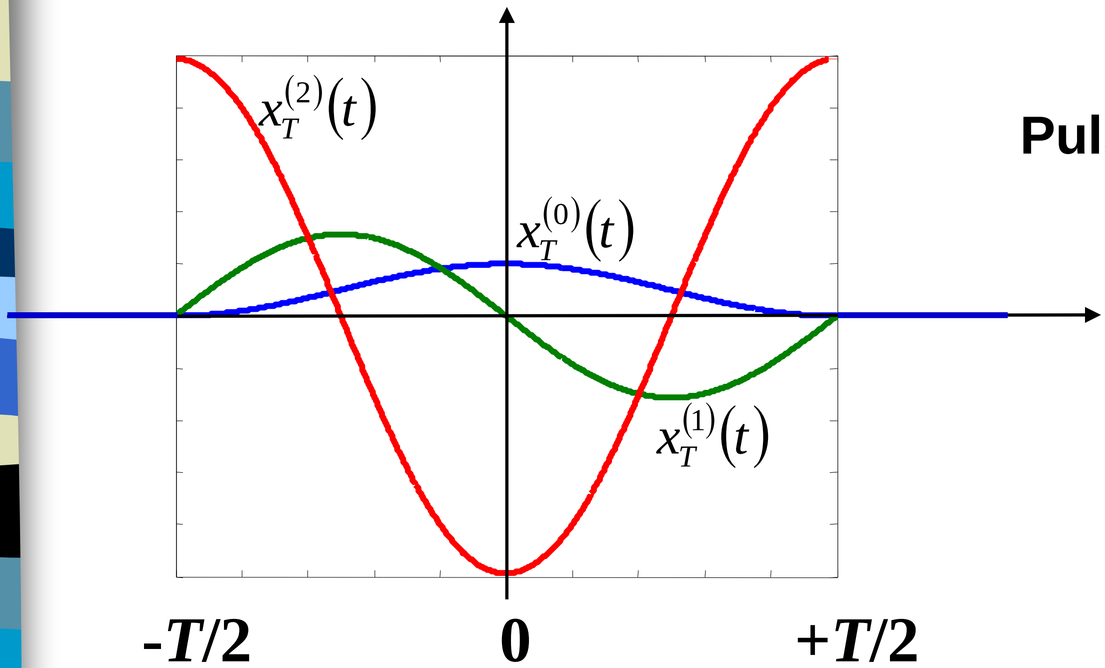




$$\Lambda_T(t) \leftrightarrow \frac{1}{2}T\mathrm{Sa}^2\frac{\omega T}{4} =$$

$$= \frac{1}{2}T\frac{\sin^2 \omega T/4}{(\omega T/4)^2} \approx \frac{1}{\omega^2},\ |\omega| \to \infty$$

Pulse shaping



Pulse „*raised cosine*" (RC_T)

$$\mathrm{RC}_T(t) = \frac{1}{2}\left(1 + \cos\frac{2\pi}{T}t\right),\ t \in [-T/2, +T/2]$$

$$\mathrm{RC}_T(\omega) = \frac{\pi^2 \sin(\omega T/2)}{\omega\left[\pi^2 - (\omega T/2)^2\right]} \sim \frac{1}{\omega^3}$$

Summary

Fourier transform decomposes a signal into a combination of harmonic signals with frequencies changing in a continuous way; amplitude and initial phase angle are determined by an a-f anf p-f characteristic, respectively.

Fourier transform features a lot of interesting properties; the most important for telecommunication purposes are:

- shift in a frequency domain (modulation),
- convolution (filtering).

Within engineering applications it is commonly assumed that signals of limited energy have a Fourier transform. Not all signals used in laboratory practice (unit step, dc component, harmonic fluctuation) meet this requirement.